

1. Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d , which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$c = \frac{\pi}{\epsilon_0} \left(\ln \frac{d}{a} \right)^{-1}$$

where a is the geometrical mean of the two radii..[15pts]

2. The two-dimensional region, $\rho \geq a$, $0 \leq \phi \leq \beta$, is bounded by conducting surfaces at $\phi = 0$, $\rho = a$, and $\phi = \beta$ held at zero potential, as indicated in Figure 2. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.

(a) Write down a solution for the potential

$\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite ρ . [10pts]

(b) Keeping only the lowest non vanishing terms, calculate the electric field components E_ρ and E_ϕ and also the surface-charge densities $\sigma(\rho, 0)$, $\sigma(\rho, \beta)$, and $\sigma(a, \phi)$ on the three boundary surfaces. [10pts]

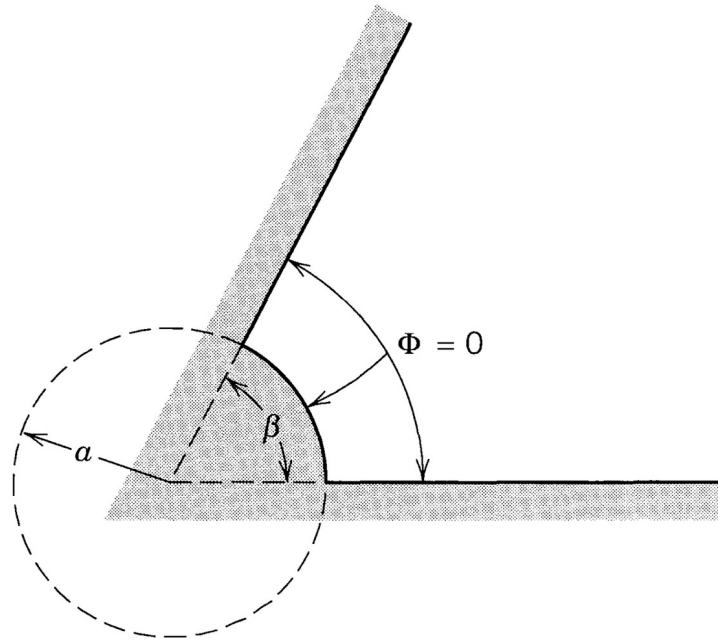


Fig. 2

3. A long cylindrical shell of radius R carries a uniform surface charge σ_0 on the upper half and an opposite charge $-\sigma_0$ on the lower half (See Fig 3). Find the electric potential inside and outside the cylinder. [15 pts]

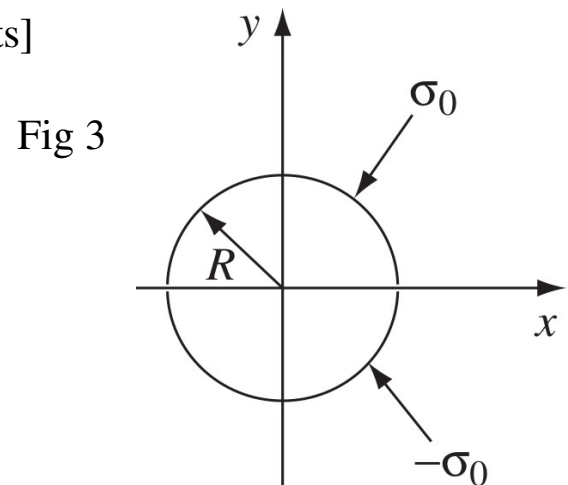


Fig 3

Classical Electrodynamics Qualifying Exam–Part II (50 points)

(1) The Biot-Savart Law and magnetostatics:

$$d\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{i d\vec{\ell} \times (\vec{x} - \vec{\ell})}{\|\vec{x} - \vec{\ell}\|^3},$$

where $\vec{\ell}$ is a source element and $\vec{x}$ is an observational point.

(a) A decrease in charges with time within volume V must correspond to a flow of charges out through the boundary ∂V . Write down the continuity equation (charge conservations).

(b) Following from part (a), write down the conditions for magnetostatics.

(c) Compute the magnetic induction $\vec{B}(R)$ due to an infinite-long, straight wire which carries currents i . Here R is the perpendicular distance to the long wire. You should use the Biot-Savart law otherwise no credits would be assigned.

(d) Starting from the Biot-Savart law, re-write $\vec{B}(\vec{x}) = \nabla \times ()$. To reach this form, first convert $i d\vec{\ell}$ to $\sum_n q_n \vec{v}_n$, insert $1 = \int d^3x' \delta^3(\vec{x} - \vec{x}')$, and then interchange the summation with the integral so that you can recognize the current density $\vec{J}(\vec{x}')$. $()$ is called gauge potential $\vec{A}(\vec{x})$. Express it in terms of some universal constants times integral which contains $\vec{J}(\vec{x}')$.

(e) Compute $[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})]^i$ to obtain the identity using the Einstein notation. Treat $\vec{A}$ as an arbitrary vector without substituting it from part (d).

• The Einstein notation means “summing over the repeated indices”. One example is

$$[\vec{\nabla} \times (\vec{\nabla} \times C)]^i = \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} \epsilon^{ijk} \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^k} C \right) \equiv \epsilon^{ijk} \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^k} C \right) = 0.$$

The quantity is zero due to the anti-symmetric tensor ϵ^{ijk} multiplied by the symmetric tensor $\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}$.

• $\epsilon^{ijk} \epsilon^{klm} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$.

(f) Compute $\vec{\nabla} \cdot \vec{A}$ by substituting $\vec{A}$ from part (d).

(g) Compute $-\nabla^2 \vec{A}$ by substituting $\vec{A}$ from part (d).

(h) Derive one of the Maxwell equation for magnetostatics by combining (d) with (e), (f) (g).

(i) Obtain the equation for the gauge potential by imposing Coulomb gauge on the Maxwell equation from part (h).

(j) Is Coulomb gauge a consistent and convenient gauge choice for the Biot-Savart Law? Give your reason for full credits.

(k) Under the following transformation

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla} \theta(\vec{x}),$$

if one would like to maintain in the same Coulomb gauge, an appropriate $\theta(\vec{x})$ must be choose. Find $\theta(\vec{x})$ in terms of an integral which contains $\vec{A}(\vec{x}')$.

(2) Materials, magnetization $\vec{M}$, magnetic induction $\vec{B}$, magnetic field $\vec{H}$, and free current density $\vec{J}$.

(a) Is the relation $\vec{B} = \mu \vec{H}$ valid for paramagnetic, diamagnetic, ferromagnetic materials? Give your reasons for full credits.

(b) Write down two of Maxwell equations for describing how materials respond to a static external magnetic induction. Also write down the relation among $\vec{B}$, $\vec{H}$ and $\vec{M}$ which is true for all different kinds of materials.

(c) When free current density $\vec{J}$ vanishes, it is much easier to solve a problem through a magnetic scalar potential Φ_M than $\vec{A}$. Use the three equations from part (b) to express a magnetic scalar potential Φ_M in terms of an integral which contains $\vec{M}(\vec{x}')$.

(d) A uniformly magnetized sphere of radius R with a given intrinsic magnetization $\vec{M}(\vec{x}') = \theta(R - r')M_0\hat{z}'$. Starting from the expression in part (c), find the following quantities inside and outside the sphere $(\Phi_M)_{\text{in}}$, $(\Phi_M)_{\text{out}}$, $\vec{B}_{\text{in}}$ and $\vec{B}_{\text{out}}$.

- $\frac{1}{||\vec{x} - \vec{x}'||} = \sum_{\ell} \sum_m \frac{4\pi}{2\ell + 1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} y_{\ell m}(\theta, \phi) y_{\ell m}^*(\theta', \phi').$
- $y_{10}(\theta', \phi') = \sqrt{\frac{3}{4\pi}} \cos(\theta').$

1. [15 points] Linearly polarized light of the form $E_x(z, t) = E_0 e^{i(kz - \omega t)}$ is incident normally onto a material which has index of refraction n_R for right-hand circularly polarized light and n_L for left-hand circularly polarized light. Use Maxwell's equations calculate the intensity and polarization of the reflected light.
2. [10 points] What is the attenuation distance for a plane wave propagating in a good conductor? Express your answer in terms of the conductivity σ , permeability μ , and frequency ω .
3. [25 points] A waveguide is made of two perfectly conducting coaxial cylinders which have inner radius r_1 and outer radius r_2 with the radiation propagating in the space between them.
 - (a) Show that it is possible to have a mode in which both the electric and magnetic fields are perpendicular to the axis of the cylinders. Describe the TEM mode. (12 points)
 - (b) Is there a cutoff frequency for this mode? (3 points)
 - (c) Calculate the velocity of propagation of this mode and the time-averaged power flow along the line. (10 points)

Classical Electrodynamics Qualifying Exam – Part I (50 points)
2024-10-18

1. [15 points] An amount of charge Q is uniformly spread out in a layer on the surface of a disc of radius R .
 - (a) Use elementary methods based on the azimuthal symmetry of the charge distribution to find the potential at any point on the axis of symmetry. (6 points)
 - (b) With the aid of (a) find an expression for the potential at any point $\vec{r}$ ($|\vec{r}| > R$) as an expansion in Legendre polynomials. (9 points)

2. [15%] A perfectly conducting sphere of radius R is placed in a uniform electric field pointing in the z -direction $\vec{E} = E_0 \hat{k}$.
 - (a) What is the surface charge density on the sphere? (8 points)
 - (b) What is the induced dipole moment of the sphere? (7 points)

3. [12%] A cylindrical capacitor has an inner conductor of radius r_1 and an outer conductor of radius r_2 . The outer conductor is grounded and the inner conductor is charged so as to have a positive potential V_0 . In terms of V_0 , r_1 , and r_2 ,
 - (a) What is the potential at r ? ($r_1 < r < r_2$) (6 points)
 - (b) If a small negative charge Q which is initially at r drifts to r_1 , by how much does the charge on the inner conductor change? (6 points)

4. [8%] Two similar charges are placed at a distance $2d$ apart. Find, approximately, the minimum radius r of a grounded conducting sphere placed midway between them that would neutralize their mutual repulsion.

- 1.** A right-circular solenoid of finite length L and radius a has N turns per unit length and carries a current I . Show that the magnetic induction on the cylinder axis in the limit $NL \rightarrow \infty$ is $B_z = \frac{\mu_0 N I}{2} (\cos \theta_1 + \cos \theta_2)$ where the angles are defined in Figure 1. [15pts]

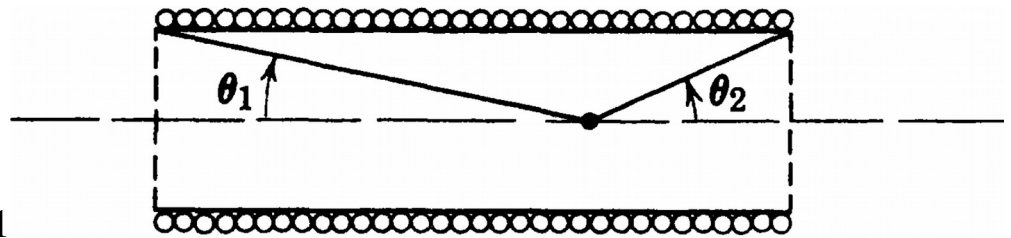


Fig 1

- 2.** A sphere of radius a carries a uniform surface-charge distribution σ . The sphere is rotated about a diameter with constant angular velocity ω . Find the vector potential and magnetic-flux density both inside and outside the sphere. [20pts]

- 3.** A sphere of linear magnetic material with μ , and of radius R , is placed in an otherwise uniform magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$, as shown in Fig 3. Find the new field $\mathbf{B}$ inside and outside the sphere. [15 pts]

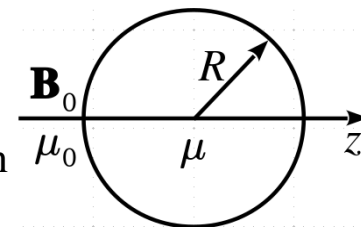


Fig 3

Classical Electrodynamics Qualifying Exam—Part III (50 points)

(1) Maxwell equations and dimensions of physical quantities which should be expressed in terms of the combinations of q (charge), m (mass), ℓ (length) and t (time).

- (a) A particle of charge q moving with velocity v in an electrical field $\vec{E}$ and a magnetic induction $\vec{B}$ experiences a “Lorentz force”. Write down the Lorentz force law.
- (b) Find the dimensions of $\vec{E}$ and $\vec{B}$ from the Lorentz force law.
- (c) Write down the relation between a time-independent magnetic field $\vec{H}$ and a static current density $\vec{J}$.
- (d) Find the dimension of $\vec{H}$ according to the relation in part (c).
- (e) Write down the relation between a time-independent electric displacement $\vec{D}$ and a static charge density ρ .
- (f) Find the dimension of $\vec{D}$ according to the relation in part (e).
- (g) Find the dimensions of electric permittivity ϵ_0 and magnetic permeability μ_0 in vacuum through the constitutive relations (the relations which connect $\vec{D}$ with $\vec{E}$ and $\vec{B}$ with $\vec{H}$).
- (h) Show that the four first-order Maxwell equations can be reduced to the two 2nd-order equations in terms of fields, $\square \vec{E} = -\mu_0 \vec{J} - \frac{\nabla \rho}{\epsilon_0}$, $\square \vec{B} = \mu_0 \vec{\nabla} \times \vec{J}$, where $\square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$.
- (i) Suppose that the fields of $\vec{E}$ and $\vec{B}$ are time dependent. Find $\vec{E}$ and $\vec{B}$ in terms of scalar potential ϕ and vector potential $\vec{A}$ using Maxwell equations.
- (j) Show that $\vec{E}$ and $\vec{B}$ are unchanged under the gauge transformations.
- (k) Show that the two 2nd-order equations in terms of potentials can be obtained from Maxwell equations when imposing Lorentz gauge, $\square \vec{A} = \mu_0 \vec{J}$, $\square \phi = \frac{\rho}{\epsilon_0}$, where $\square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$.

(2) A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\vec{\nabla} \Phi$. In this region, it also exists a small localized time-independent current density $\vec{J}(\vec{x})$, which generates a magnetic field $\vec{H}$.

- (a) Show that the momentum of these EM fields, $\vec{P} \equiv \int d^3x [\vec{D} \times \vec{B}]$, can be transformed to $\vec{P} = \frac{1}{c^2} \int d^3x \Phi \vec{J}$ by dropping the surface term.
 - $\epsilon^{ijk} \left(\frac{\partial}{\partial x^j} A \right) B^k = \epsilon^{ijk} \frac{\partial}{\partial x^j} [AB^k] - \epsilon^{ijk} A \frac{\partial}{\partial x^j} B^k$.
- (b) Justify in which situation you can ignore the surface term in part (a).
- (c) Assuming that the current distribution is localized to a region small compared to the scale of the variation of the electric field, that is, using $\Phi = \Phi_0 - \vec{E}_0 \cdot \vec{x}$. Show $\vec{P} = \frac{1}{c^2} \vec{E}_0 \times \vec{m}$, where $\vec{m}$ is a magnetic dipole moment.
 - localized source $\implies \int d^3x J^i(\vec{x}) = 0$, $\int d^3x [x^k J^\ell(\vec{x}) + x^\ell J^k(\vec{x})] = 0$.
 - $\epsilon^{ijk} \epsilon^{klm} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$.

(d) Suppose the current distribution is placed instead in a uniform electric field $\vec{E}_0$ (filling all space). Show that the result in (c) is changed by a surface integral contribution, yielding to $\vec{P} = \frac{2}{3c^2} \vec{E}_0 \times \vec{m}$.

- $\int d\Omega \hat{r}^i = 0$, $\int d\Omega \hat{r}^i \hat{r}^j = \frac{4\pi}{3} \delta^{ij}$, $\int d\Omega \hat{r}^i \hat{r}^j \hat{r}^k = 0$, $\vec{H}_{\text{dipole}} = \frac{1}{4\pi} \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{r^3} + \frac{2}{3} \vec{m} \delta^3(\vec{r})$.

Classical Electrodynamics Qualifying Exam – Part I (50 points)
2024-03-15

1. [12 points] A static electric charge is distributed in a spherical shell of inner radius R_1 and outer radius R_2 . The electric charge density is given by $\rho(r) = a + br$, where r is the distance from the center, and zero everywhere else.
 - (a) Find an expression for the electric field everywhere in terms of r . (6 points)
 - (b) Find expressions for the electric potential and energy density for $r < R_1$. Take the potential to be zero at $r \rightarrow \infty$. (6 points)
2. [13%] Find the potential energy of a point charge Q in vacuum a distance d away from a semi-infinite dielectric medium whose dielectric constant is κ . (13 points)
3. [25%] If the potential of a spherical shell with radius R of zero thickness depends only on the polar angle θ and is given by $V(\theta)$, inside and outside the sphere there being empty space,
 - (a) Show how to obtain expressions for the potential $V(r, \theta)$ inside and outside the sphere and how to obtain an expression for the electric sources on the sphere. (15 points)
 - (b) Solve with $V(\theta) = V_0 \cos^2 \theta$. (10 points)

The first few of the Legendre polynomials are given as follows:

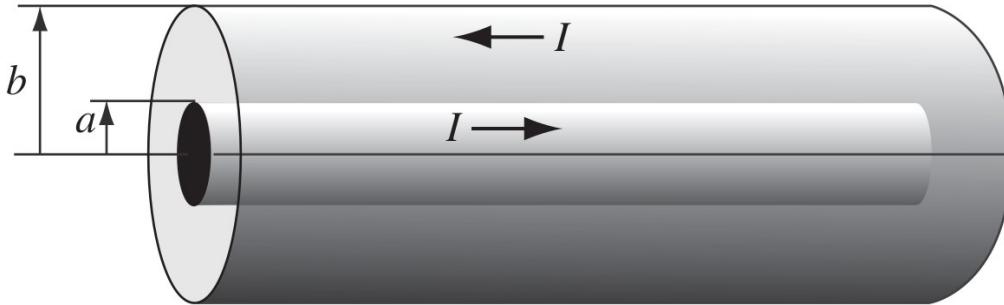
$$P_0(\cos \theta) = 1, \quad P_1(\cos \theta) = \cos \theta, \quad P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1)$$

We also have

$$\int_0^\pi P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2\delta_{nm}}{2n+1}.$$

- 1.** A long coaxial cable carries current I (the current flows down uniformly inside the inner cylinder of radius a , and back along the surface of the outer cylinder of radius b) as in Fig. I. Find the magnetic energy in a section of length ℓ . [15pt]

Fig 1

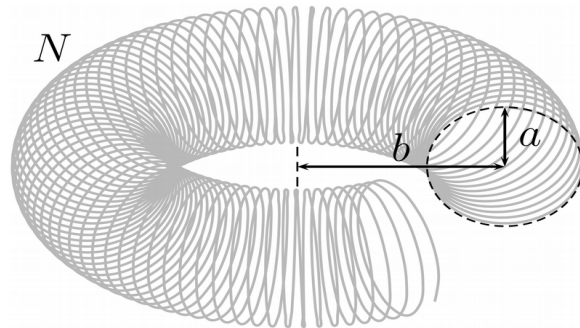


- 2.** Consider an infinitely long cylindrical conductor of radius a , with a constant current I flowing in uniformly.

- (a) Find the magnetic vector potential [10pts]
 (b) and the corresponding magnetic field [10pts]

- 3.** A toroidal coil has N turns, a mean radius b , and a circular cross section of radius a , as shown in Fig III. Show that its inductance is $L = \mu_0 N^2 (b - \sqrt{b^2 - a^2})$. [15 pts]

Hint: $\int_0^\pi \frac{\sin^2 \phi \, d\phi}{1 + \beta \cos \phi} = \frac{\pi}{\beta^2} (1 - \sqrt{1 - \beta^2})$



1. [25 pts] The electric field of an electromagnetic wave in free space has the following form:

$$\mathbf{E}(\mathbf{r}, t) = 10(\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}) \cos(\omega t + 3x - y - z) \text{ (V/m)}$$

- (a) [10 pts] Determine the direction of propagation and the values E_z and ω .
 (b) [5 pts] Calculate the magnetic field $\mathbf{H}(\mathbf{r}, t)$.
 (c) [10 pts] Calculate the Poynting vector $\mathbf{S}$.
2. [15 pts] From the Maxwell equations with sources ($\rho \neq 0, \mathbf{J} \neq 0$), we can define:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

- (a) [10 pts] Derive the wave equations for the scalar potential $\phi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ in the Lorenz condition: $\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial \phi}{\partial t}$.
 (b) [5 pts] Write the gauge transformation: $\phi(\mathbf{r}, t) \rightarrow \phi'(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}'(\mathbf{r}, t)$ such that $\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}'(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t)$.
3. [10 pts] Let θ_i be the angle of incidence of a wave from a medium with index of refraction n to a medium with n' . For polarization of the wave perpendicular to the plane of incidence, the reflection coefficient R reads:

$$R = \left| \frac{n \cos \theta_i - n' \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n \cos \theta_i + n' \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2;$$

while for polarization parallel to the plane of incidence, the coefficient reads:

$$R = \left| \frac{n' \cos \theta_i - n \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n' \cos \theta_i + n \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2.$$

- (a) [5 pts] Determine the Brewster's angle θ_B for a total transmission of wave energy.
 (b) [5 pts] Determine the critical angle θ_C for a total internal reflection.

<i>Useful Identity</i>

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \text{ for an arbitrary vector } \mathbf{V}.$$

Classical Electrodynamics Qualifying Exam—Part I (50 points)

(1) Consider a point charge q located at position $\vec{y}$ **inside** a hollow, grounded, conducting sphere of radius a [25pts].

(a) Write down the potential and Green function inside the sphere. You do not need to derive it.

(b) Compute the induced surface charge density.

(c) Compute the induced charge on the sphere from part (b).

(d) Find the force acting on the point charge q (including magnitude and direction).

(e) If the boundary condition is changed from the grounded sphere to having a fixed potential V on the sphere, what are the change of the potential $\Delta\Phi$, the change of the surface charge density $\Delta\sigma$ and the change of the force $\Delta\vec{F}$ on the point charge q ?

(2) The potential on the $z=0$ plane is specified to be $\Phi=V$ inside a circle of radius a centered at the origin, and $\Phi=0$ outside a circle [25pts].

(a) Write down the Green function with Dirichlet boundary conditions for $z \geq 0$ (express the final result in terms of the observational point coordinates $[\rho, \phi, z]$ and the (image) charge coordinates $[\rho', \phi', z']$).

(b) Find an integral expression for the potential at points $z \geq 0$ in terms of cylindrical coordinates (simplify the integral without performing it at this stage).

(c) Find the potential when the observation point on the $\hat{z}$ axis ($\rho=0$).

1. A long circular cylinder of radius R carries a magnetization $\mathbf{M} = k s^2 \hat{\phi}$ where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector (Fig 1). Find the magnetic field due to $\mathbf{M}$, for points inside and outside the cylinder. [20 pts]

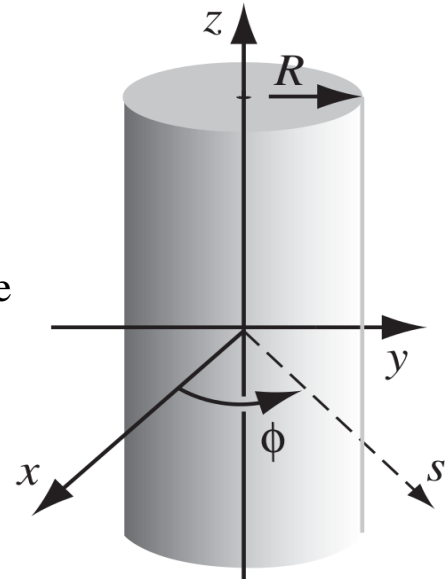


Fig 1

2. A cylindrical rod with radius b carries uniform axial current density $\mathbf{J} = J \hat{\mathbf{z}}$. A cylindrical cavity with radius a is hollowed out from the rod at an arbitrary location, as shown in Fig. 2. Assume that the current density in the remaining part stays the same. Let $\mathbf{d}$ be the position of the center of the cavity with respect to the center of the rod. Show that the magnetic field inside the cylindrical cavity is uniform in both magnitude and direction. [15 pts]

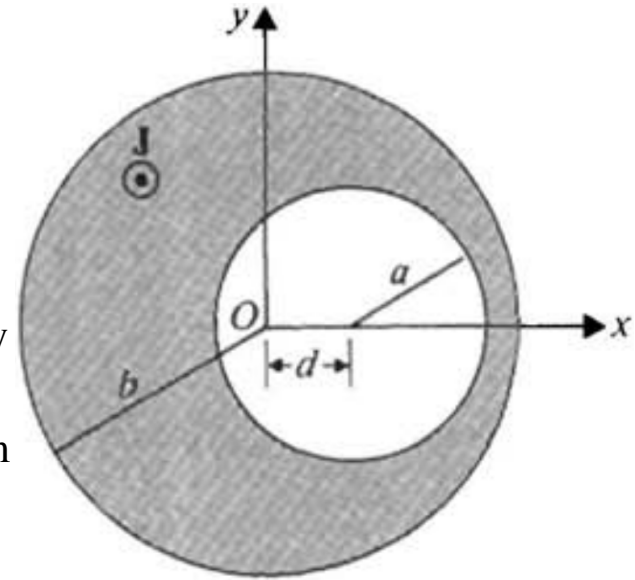


Fig 2

3. Suppose $\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \Theta(vt - r) \hat{\mathbf{r}}$, $\mathbf{B}(\mathbf{r}, 0) = 0$, where $\Theta(x) \equiv \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$, the step function, and $\frac{d\Theta}{dx} = \delta(x)$. Show that these fields satisfy all of Maxwell's equations, and determine the charge density $\rho(\mathbf{r}, t)$ and the current density $\mathbf{J}(\mathbf{r}, t)$. Describe the physical situation that gives rise to these fields. [15 pts]

Classical Electrodynamics Qualifying Exam—Part III (50 points)

(1) Transformation Properties of Electromagnetic Fields [15pts].

(a) Determine how polarization $\vec{P}$, electric field $\vec{E}$, magnetic induction $\vec{B}$ transform under parity, and time reversal.

(b) Use spatial rotation, parity and time reversal to build the allowed model of polarization $\frac{1}{\epsilon_0}\vec{P}$ which consists of $\vec{E}$ and $\vec{B}$ up to second time derivatives in total, maximally quadratic in fields and no spatial derivatives involved (assume that the proportional scalar coefficients are invariant under all transformations).

(2) Suppose a plane (interface) divides the whole space into half-infinite regions, medium 1 and medium 2. A plane EM wave of angular frequency ω propagates from the medium 1 (permeability μ , index of refraction n) with the angle of incidence i to the medium 2 (permeability μ' , index of refraction n'). If the electric field polarization is perpendicular to the plane of incidence, what fraction of the incident wave (electric field) is reflected [15pts]? Is it possible to choose a special angle to make reflection vanish (you can make a following approximation here, $\mu \sim \mu'$) [5pts]?

• You need to use the relation between $\vec{E}$ and $\vec{B}$ of a plane wave, snell's law, and the two boundary conditions at the interface following from $\vec{\nabla} \times \vec{E} = -\vec{\dot{B}}$, $\vec{\nabla} \times \vec{H} = \epsilon_0 \vec{\dot{E}}$.

(3) Consider a plasma of elections of density N_e (numbers per volume) under the assumption that each electron feels the combination of the Lorentz force and the Abraham-Lorentz force,

$$\vec{F}_{\text{AL}} = \frac{2e^2}{3\epsilon_0 c^3} \left(\frac{d}{dt} \right)^3 \vec{x}(t).$$

What is the electric permittivity $\epsilon(\omega)$ in the low frequency domain [15pts]?

• “low frequency domain” means that you can ignore the spatial variation in the phase of a plane wave. You also can simplify the equation of motion by dropping one of a smaller term from the Lorentz force contribution.

Classical Electrodynamics Qualifying Exam – Part I (50 points)
2023-03-17

1. [20 points] An amount of charge q is uniformly spread out in a layer on the surface of a disc of radius R .
 - (a) Use elementary methods based on the azimuthal symmetry of the charge distribution to find the potential at any point on the axis of symmetry. (5 points)
 - (b) With the aid of (a) find an expression for the potential at any point $\vec{r}$ ($|\vec{r}| > R$) as an expression in angular harmonics. (15 points)

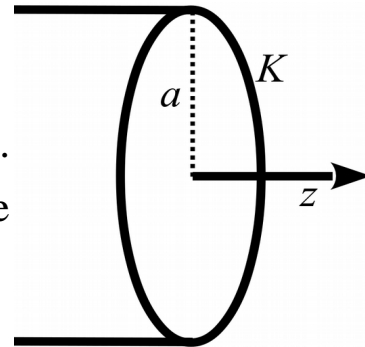
2. [20%] A thin metallic sphere of radius R has charge Q .
 - (a) What is the capacitance? (5 points)
 - (b) What is the energy density of the electric field at a distance r ($r > R$) from the sphere's center? (5 points)
 - (c) What is the total energy of the field? (5 points)
 - (d) A potential V is established between the inner (radius R_1) and outer (radius R) concentric thin metal spheres. What is the radius of the inner sphere such that the electric field near its surface is a minimum? (5 points)

3. [10%] Consider two electric dipoles $\vec{P}_1$ and $\vec{P}_2$ separated by a distance d . Find the force between them due to the electrostatic interaction between the two dipole moments, for arbitrary orientation of the direction of $\vec{P}_1$ and $\vec{P}_2$.

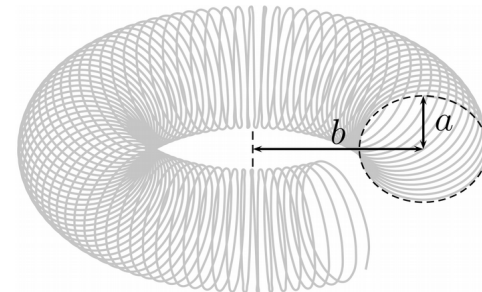
1. (a) Find the vector potential a distance s from an infinite straight wire carrying a current I . Check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. [10 pt]
 (b) Find the magnetic potential inside the wire, if it has radius R and the current is uniformly distributed. [10 pts]

2. Consider an uniform surface current $K \hat{\mathbf{z}}$ over the (infinitely-long) cylindrical surface $s = a$. Assume that the interior and exterior space are both vacuum and the vacuum magnetic susceptibility is μ_0 .

- (a) What is the direction of the magnetic force exerted on the surface current? (in terms of $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$, or their linear combination.) [5pts]
 (b) Find the magnetic field $\mathbf{B}$ for $s > a$ and $s < a$. [5pts]
 (c) Show that the magnitude of the magnetic force per unit area exerted on the surface current is $\frac{1}{2} \mu_0 K^2$ [10 pts]



3. A toroidal coil has N turns, a mean radius b , and a circular cross section of radius a . Show that its inductance per unit length is $L = \mu_0 N^2 (b - \sqrt{b^2 - a^2})$. [10 pts]



Qualifying Exam: Electrodynamics

Part III (50 Points)

1. [15pts] Consider the total electric charge density $\rho \neq 0$ and the total electric charge current $\mathbf{J} \neq 0$.
- (a) [5pts] Write down the Maxwell equations in differential form.
 - (b) [3pts] Write down the continuity equation for ρ and $\mathbf{J}$.
 - (c) [7pts] Define the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$ as follows

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} ,$$

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

A new set of potentials ϕ' and $\mathbf{A}'$ may be used to define $\mathbf{E}$ and $\mathbf{B}$. Write down the equations that relate the two sets of potentials?

2. [15 pts] The electric field of an electromagnetic wave has the form:

$$\mathbf{E} = E_0 \hat{\mathbf{e}}_x \cos(\omega\sqrt{\mu_0\epsilon_0}z - \omega t) .$$

- (a) [10 pts] Find the magnetic field strength $\mathbf{H}$ of the electromagnetic wave.
 - (b) [5 pts] Find the Poynting vector $\mathbf{S}$ or the energy flux and its time average over a period T .
3. [20 pts] For transverse magnetic (TM) waves and transverse electric (TE) waves propagating in a rectangular waveguide with inner dimensions $a \times b$ (see Figure 1), find:
- (a) [5pts] the cutoff wavenumber k_c ,
 - (b) [5pts] the cutoff frequency f_c ,
 - (c) [5pts] the cutoff wavelength λ_c ,
 - (d) [5pts] the phase velocity v_p .



Figure 1: A rectangular waveguide.

Classical Electrodynamics Qualifying Exam–Part I (50 points)

(1) *Dimension (15 points)*: Express your results in terms of fundamental quantities—time “ t ”, length “ ℓ ”, mass “ m ” and charge “ q ”.

(a) Find dimensions of the electric field $\vec{E}$ and the magnetic induction $\vec{B}$ by using the Lorentz force law,

$$\vec{F} = q [\vec{E} + \vec{v} \times \vec{B}] .$$

(b) Find dimensions of the electric displacement $\vec{D}$ and the magnetic field $\vec{H}$ by using some of Maxwell equations.

(c) Find dimensions of the electric permittivity ϵ and the magnetic permeability μ by using their respective constitutive relations,

$$\vec{D} = \epsilon \vec{E} \quad , \quad \vec{B} = \mu \vec{H} .$$

(2) *Method of Images (20 points)*: Consider a point charge q located at position $\vec{y}$ inside a hollow, grounded, conducting sphere of radius a .

(a) Find the image charge and its location (you do not need to derive them).

(b) Find the Dirichlet Green function $G_D(x; x')$ for the space inside a sphere.

(c) Compute the induced surface charge density σ .

(d) Compute the induced charge on the sphere.

(e) Find the force $\vec{F}$ acting on the point charge q (including magnitude and direction).

(f) If the boundary condition is changed from the grounded sphere to having a total charge Q on the sphere, what are the change of the potential $\Delta\Phi$, the change of the surface charge density $\Delta\sigma$ and the change of the force $\Delta\vec{F}$ on the point charge q ?

(g) Find $\Delta\Phi$, $\Delta\sigma$ and $\Delta\vec{F}$ if the boundary condition is changed from the grounded sphere to having a fixed potential V on the sphere.

(3) *Green Functions (15 points)*: The Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions in a square two-dimensional region, $0 \leq x \leq 1$, $0 \leq y \leq 1$ has an expansion,

$$G(x, y; x', y') = \sum_{n=1}^{n=\infty} u_n(x) u_n^*(x') g_n(y, y') , \quad u_n(x) \text{ is an appropriate orthogonal basis .}$$

(a) Show $g_n(y, y')$ obeys the following equation,

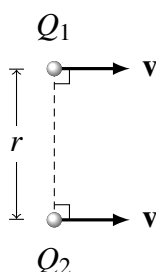
$$\left[\frac{\partial^2}{\partial y^2} - (n\pi)^2 \right] g_n(y, y') = -4\pi \delta(y - y') .$$

(b) Solve for $g_n(y, y')$ using appropriate boundary conditions and obtain $G(x, y; x', y')$.

Classical Electrodynamics

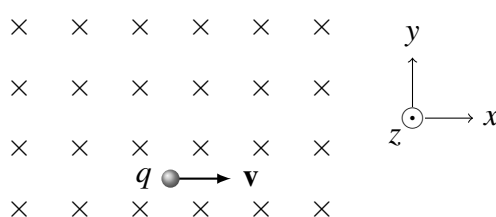
Part II (50 points)

- (10 points)** What is Lorentz force? Write down the equation and explain the meaning of the symbols.
- (10 points)** What is Biot-Savart law? Apply the law to find the magnetic field for a particular case at your own choice.
- (10 points)** What is Ampère's law? Apply the law to find the magnetic field for a particular case at your own choice.
- (10 points)** Two point charges Q_1 and Q_2 move side by side at the same velocity in vacuum, as sketched below:



Find the magnitude of the force between Q_1 and Q_2 .

- A charged particle moves at an initial velocity $\mathbf{v} = (v, 0, 0)$ in a constant magnetic field $\mathbf{B} = (0, 0, -B)$ with $B > 0$, as sketched below:



- (1 point)** If $q > 0$, make a qualitative sketch for the trajectory of the charged particle.
- (1 point)** Do the same as (a) but with $q < 0$.
- (4 points)** If a constant electric field $\mathbf{E} = (E, 0, 0)$ with $E > 0$ is applied in addition to the existing $\mathbf{B}$ field, make a qualitative sketch for the trajectory of the particle, considering $q > 0$.
- (4 points)** Do the same as (c) but considering $q < 0$.

Electrodynamics (Part III, 50 points) 2022-10-14

1. The Maxwell equations can be rephrased as two equations,

$$\begin{cases} \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0} \rho \\ \left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \mathbf{J} \end{cases}, \dots\dots\dots(*)$$

which contains all the information in Maxwell's equations.

- (a) What is the gauge transformation? **(10 points)**
- (b) Show that in the Lorentz gauge the addressed two equations (*) can be rephrased in the form of two independent inhomogeneous wave equations. **(10 points)**
2. (a) Write down Maxwell's equations in media. **(5 points)**
- (b) Use Maxwell's equations, derive the boundary conditions for E and B fields of electromagnetic waves across the boundary between two linear media (electric permittivity: ϵ_1, ϵ_2 , electric permeability: μ_1, μ_2 , respectively). **(5 points)**
3. A monochromatic electromagnetic plane wave is propagating toward the z-direction in a conductor, where the polarization of the electromagnetic plane wave is along x-direction. The wave number of the plane wave in a conductor is a complex number, i.e., $\tilde{k} = k + i\kappa$.
- (a) Determine the amplitude and the phase relation between the E-field and the B-field of the plane wave. **(10 points)**
- (b) Express B-field of the plane wave by the E-field amplitude E_0 and E-field phase δ_E . **(10 points)**

1. [15 points] A point charge q is at the center of an uncharged spherical conducting shell, of inner radius a and outer radius b . How much work would it take to move the charge out to infinity (through a tiny hole drilled in the shell)?

2. [10 points] Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r . (Hint: $\nabla \cdot \vec{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta V_\theta) + \frac{1}{r \sin \theta} \frac{\partial V_\phi}{\partial \phi}$)

3. [15 points] Prove the uniqueness theorem: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S .

4. [10 points] A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 1). Find the potential outside the sphere.

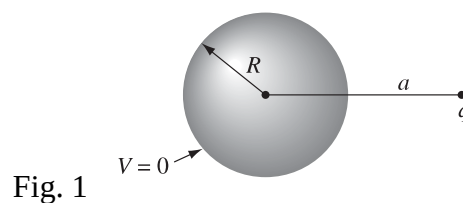


Fig. 1

Classical Electrodynamics (part II, 50 points)

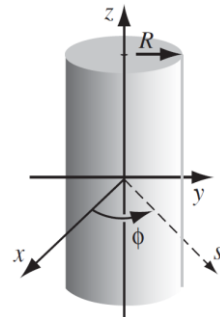
1. (a) The physical meaning of continuity equation $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$. (3%) (b) Does Ampere law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ satisfy continuity equation? (5%)
2. A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a , and back along the outer cylinder, radius b) as shown in below Figure. (a) Find the magnetic field $\vec{B}$ between the cylinders. (6%) (b) Find the magnetic energy W stored in the section of length l . (6%)



Hint: Ampere's law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$

Hint: Energy in magnetic field $W = \frac{1}{2\mu_0} \int B^2 d\tau$ (all space)

3. A long circular cylinder of radius R carries a magnetization $\vec{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector. (a) Find the volume bound current $\vec{J}_b$. (6%) (b) Find the surface bound current $\vec{K}_b$. (6%) (c) Find the total current I . (6%) (d) Find the magnetic field $\vec{B}$ inside the cylinder. (6%) (e) Find the magnetic field $\vec{B}$ outside the cylinder. (6%)



Hint:

volume bound current $\vec{J}_b = \vec{\nabla} \times \vec{M}$

surface bound current $\vec{K}_b = \vec{M} \times \vec{n}$

total current $I = \int \vec{J} \cdot d\vec{a}$, where $\vec{J} = \vec{J}_b + \vec{K}_b$, and $d\vec{a} = s ds d\phi \hat{z}$

Qualifying Exam: Electrodynamics

Part 3 (50 Points)

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1. [5 points] An electric field has the form:

$$\mathbf{E} = \hat{\mathbf{e}}_x 20 e^{i\pi/4} e^{-2\pi i y} + \hat{\mathbf{e}}_z E_0 e^{i\varphi} e^{-2\pi i y}.$$

Determine the types of polarization of the wave for the following values of E_0 and φ :

- (a) [2 points] $E_0 = 20, \varphi = 5\pi/4$;
 (b) [3 points] $E_0 = 20, \varphi = 3\pi/4$.
2. [5 points] The dispersion relation for electromagnetic waves in a certain medium is given by $\omega^2 = ak$, where a is a constant, ω the frequency and k the magnitude of the wave vector. Find the velocity of energy propagation v_g and the phase velocity v_p in this medium.
3. [20 points] Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0 , frequency ω , and phase angle zero that is
- (a) [10 points] traveling in the negative x direction and polarized in the positive z direction;
 (b) [10 points] traveling in the direction from the origin to the point $(1, 1, 1)$, with polarization parallel to the xz plane.

In each case, give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$.

4. [20 points] Consider an electromagnetic wave propagating in the positive z direction. The wave is polarized in the x direction, has a frequency of 1 GHz, and an amplitude of 100 V/m. The wave has an initial phase of $\varphi = 0$ and is incident normally from a medium with ϵ_0 and μ_0 into a non-dissipative medium with ϵ_1 and μ_0 .

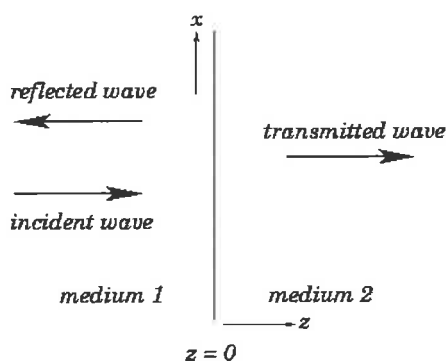


Figure 1:

- (a) [10 points] Write down the electric and magnetic fields in the media.
 (b) [10 points] Find the reflection coefficient R and transmission coefficient T .

Classical Electrodynamics Part I (50 points)

Useful relations:

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\int_a^0 \frac{\sqrt{x}}{\sqrt{d-x}} dx = -d \frac{\pi}{2}$$

$$\begin{aligned} \nabla \cdot \vec{V} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta V_\theta) \\ &\quad + \frac{1}{r \sin\theta} \frac{\partial V_\phi}{\partial \phi} \end{aligned}$$

1. [15 points] The potential generated by a dipole $\vec{p}$ is given as

$$V_{dip}(r, \theta) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}.$$

Derive the electric field $\vec{E}$ and write it in coordinate-free form.

2. [15 points] A point charge q of mass m is released from a rest at a distance d from an infinite grounded conducting plane. How long will it take for the charge to hit the plane?

3. [20 points] The electric potential of some configuration is given as:

$$V(\vec{r}) = A \frac{\exp[-\lambda r]}{r},$$

where A and λ are constants. Find the electric field $\vec{E}(\vec{r})$, the charge density $\rho(r)$, and the total charge Q .

Classical Electrodynamics

Part II (50 points)

1. The Biot-Savart law reads

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{\ell} \times \mathbf{r}}{|\mathbf{r}|^3}, \quad (1)$$

where μ_0 is the free-space permeability.

- (5 points)** Explain Eq. (1) briefly with a proper sketch.
 - (7 points)** Consider a point charge Q moving at velocity $\mathbf{v}$, as sketched in Fig. 1(a). Find the magnetic field $\mathbf{B}$ at an arbitrary point P located at $\mathbf{r}$ relative to the point charge, using the Biot-Savart law (1). Express your result in terms of the speed of light in vacuum, $c = 1/\sqrt{\mu_0\epsilon_0}$, where ϵ_0 is the free-space permittivity, the electrostatic field $\mathbf{E}$ due to the point charge Q , and the velocity $\mathbf{v}$.
 - (8 points)** Two point charges Q_1 and Q_2 move side by side at the same velocity in vacuum, as sketched in Fig. 1(b). Apply the result found in 1b to find the magnitude of the force between Q_1 and Q_2 .
2. Consider a current density function $\mathbf{J}$ in free space; see Fig. 1(c). Complete the following tasks. Note that “primed” and “unprimed” coordinates and differential operators should be clearly and carefully distinguished.
- (4 points)** Show that the magnetic field at position $\mathbf{r}$ due to $\mathbf{J}$ can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r', \quad (2)$$

based on the Biot-Savart law (1).

- (3 points)** Prove

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\vec{\nabla} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

and use it to show that Eq. (2) can be rewritten as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (3)$$

- (3 points)** Show that Eq. (3) is divergenceless, i.e., $\vec{\nabla} \cdot \mathbf{B} = 0$.
- (11 points)** Calculate $\vec{\nabla} \times \mathbf{B}$ based on Eq. (3). Hint: There should be two terms left. Furthermore, the following identities may be directly used:

$$\vec{\nabla} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\vec{\nabla}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad \nabla'^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

- (3 points)** Write down the equation of continuity, and explain it briefly.
- (2 points)** Apply the equation of continuity at steady-state to the result of 2d.
- (4 points)** Use the result of 2f to derive Ampère’s law.

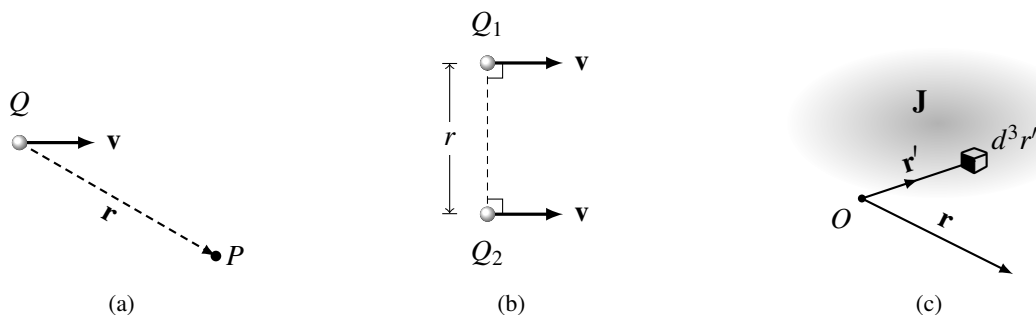


Figure 1: Sketches for (a) problem 1b, (b) problem 1c, and (c) problem 2a.

Qualifying Exam: Electrodynamics

Part 3: 50%

1. [20%] An electric field of an electromagnetic wave in free space has the following form:

$$\mathbf{E}(\mathbf{r}, t) = 10(\hat{\mathbf{e}}_x + 2\hat{\mathbf{e}}_y + E_x\hat{\mathbf{e}}_z) \cos(\omega t + 3x - y - z) \quad (\text{V/m})$$

- (a) [10%] Determine the direction of propagation and the values E_x and ω .
- (b) [10%] What is the time-average energy flow?
2. [10%] A plane wave in free space propagating in the $+z$ direction is

$$\mathbf{E} = \hat{\mathbf{e}}_x 100 \sin(\omega t - kz) + \hat{\mathbf{e}}_y 200 \cos(\omega t - kz) \quad (\text{V/m})$$

The wave is incident on an infinite conducting xy plane at $z = 0$. What is the reflected wave $\mathbf{E}_r$?

3. [10%] Let θ_i be the angle of incidence of a wave from a medium with index of refraction n to a medium with n' . For the polarization of the wave perpendicular to the plane of incidence, the reflection coefficient R reads:

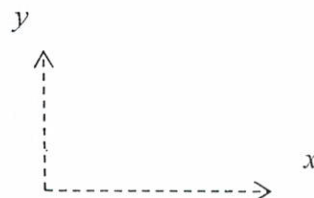
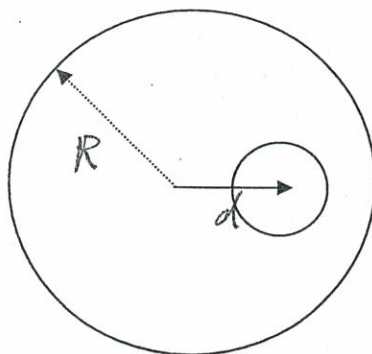
$$R = \left| \frac{n \cos \theta_i - n' \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n \cos \theta_i + n' \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2,$$

while for polarization parallel to the plane of incidence, the coefficient reads:

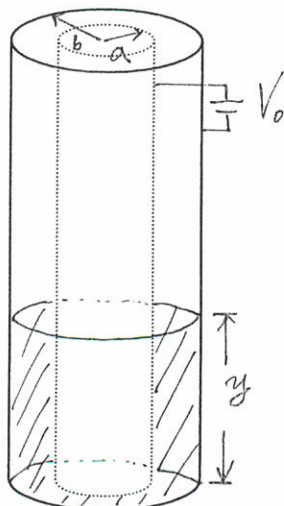
$$R = \left| \frac{n' \cos \theta_i - n \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n' \cos \theta_i + n \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2.$$

- (a) [5%] Determine Brewster's angle θ_B for a total transmission of wave energy.
- (b) [5%] Determine critical angle θ_C for a total internal reflection.
4. [10%] Let $\phi(\mathbf{r}, t)$ be a scalar potential and $\mathbf{A}(\mathbf{r}, t)$ a vector potential.
- (a) [5%] Write down the electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic fields $\mathbf{B}(\mathbf{r}, t)$ in terms of $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$.
- (b) [5%] Write down the gauge transformation for $\phi(\mathbf{r}, t) \longrightarrow \phi'(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t) \longrightarrow \mathbf{A}'(\mathbf{r}, t)$.
-

1. (10 Points) Find the magnitude and direction of the electric field $\vec{E}$ in the spherical cavity whose center is displaced from that of the solid part by a distance d . The uniform charge density is ρ_0



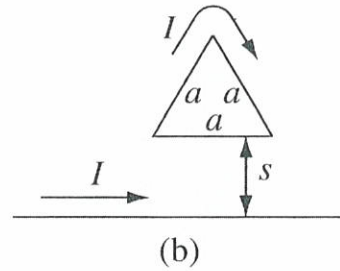
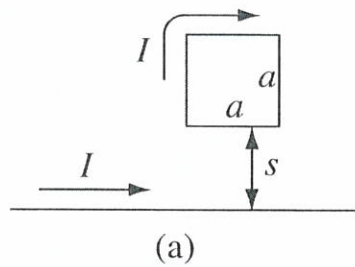
2. (15 Points) A point charge q is embedded at the center of a sphere of linear dielectric material (electric permittivity ϵ and radius R). (a) Find the electric displacement $\vec{D}$, the electric field $\vec{E}$, as well as the polarization $\vec{P}$ inside the sphere. (b) What is the polarization surface charge density?
3. (10 Points) An electric dipole $\vec{p}$ with a magnitude of 2×10^{-27} C-m is initially located at an origin along the z -axis. (a) Determine the rotational torque on this dipole when an electric field $\vec{E} = 5\hat{x} + 10\hat{y}$ (V/m) is applied. (b) How much energy changes if $\vec{p}$ finally points to the direction of $\vec{E}$?
4. (15 Points) A long coaxial cylindrical capacitor of inner radius a and outer radius $b = 2a$ stands vertically in a tank of dielectric oil (electric permittivity ϵ and mass density ρ_m). The inner one is maintained at potential V_0 , while the outer one is grounded. To what height y does the oil rise in the space between a and b .



Classical Electrodynamics Part II (50 points)

1. [10 points] (a) Find the force on a square loop placed as shown in Fig. 1(a), near an infinite straight wire. Both the loop and the wire carry a steady current I .

[10 points] (b) Find the force on the triangular loop in Fig. 1(b).



2. [10 points] What current density would produce the vector potential, $\mathbf{A} = k\hat{\phi}$ (where k is a constant), in cylindrical coordinates?
3. [10 points] (a) Starting from the Lorentz force law, in the form as:

$$\vec{F} = \int I(d\vec{\ell} \times \vec{B}),$$

Find that the torque on *any* steady current distribution (not just a square loop) in a uniform field is $\vec{m} \times \vec{B}$, where $\vec{m}$ is the magnetic dipole moment of a loop.

[10 points] (b) A uniform current density $\mathbf{J} = J_0 \hat{z}$ fills a slab straddling the yz plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0 \hat{x}$ is situated at the origin. Find the force on the dipole using $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$.

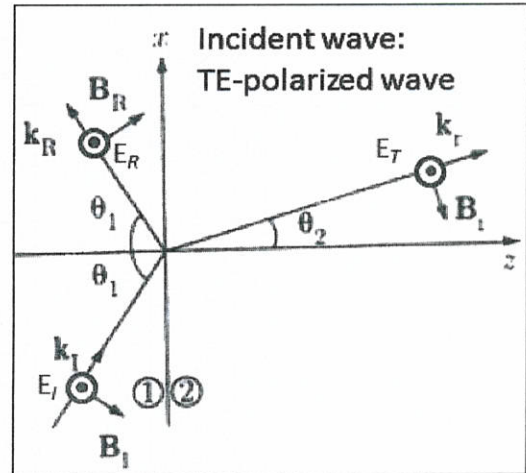
Electrodynamics (Part III, 50 points) 2020-03-27

1. Assuming that a monochromatic point light source is spreading out into every direction.

(a) Please write down the wave function of the wave and show that your solution satisfies the wave equation. (15 points)

(b) What is the expression for the Poynting vector of the wave? Find the momentum density of this wave (10 points)

2. Referring to the figure, an electromagnetic wave of polarization perpendicular to the plane of incidence is obliquely impinging on the boundary. Medium 1 and 2 are both dielectrics. The incident, reflected, transmitted waves are represented as following:

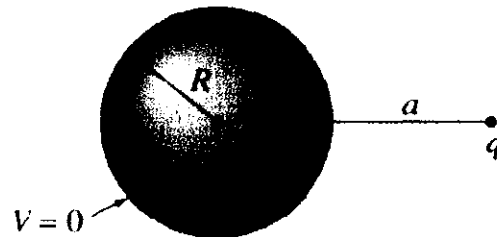


$$\begin{cases} \tilde{\mathbf{E}}_I = \tilde{E}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I = \frac{1}{v_1} \tilde{E}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} (-\cos \theta_1 \hat{\mathbf{x}} + \sin \theta_1 \hat{\mathbf{z}}); \\ \tilde{\mathbf{E}}_R = \tilde{E}_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} (\cos \theta_1 \hat{\mathbf{x}} + \sin \theta_1 \hat{\mathbf{z}}); \\ \tilde{\mathbf{E}}_T = \tilde{E}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} (-\cos \theta_2 \hat{\mathbf{x}} + \sin \theta_2 \hat{\mathbf{z}}); \end{cases}$$

- (a) Write down boundary conditions of E-field and B-field. (10 points)
- (b) Impose the boundary conditions and obtain the Fresnel equations for E_{0R} and E_{0T} . (15 points)

Electrodynamics, Part I (50 points)

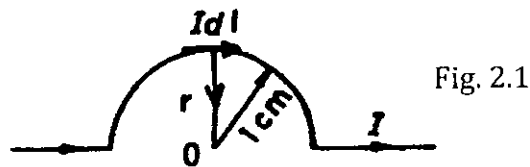
1. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R .
 - (a) Find the potential outside the sphere. (10 points)
 - (b) Find the induced surface charge distribution on the sphere, and the total induced charge on the sphere. (10 points)
 - (c) Calculate the energy of this configuration. (10 points)



2. A dielectric sphere having a radius a and a uniform linear dielectric constant ϵ_r is placed in an otherwise uniform electric field E_0 . Find:
 - (a) the potential inside the sphere, (5 points)
 - (b) the potential outside the sphere, (5 points)
 - (b) the electric field inside the sphere, and (5 points)
 - (c) the polarization-surface-charge density. (5 points)

Classical Electrodynamics Part II (50 points)

1. [10 points] As shown in Fig. 2.1, an infinitely long wire carries a current $I = 1$ A. It is bent so as to have a semi-circular detour around the origin, with radius 1 cm. Calculate the magnetic field at the origin. [permeability of free space $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$]



2. [15 points] A cylindrical wire of permeability μ carries a steady current I . If the radius of the wire is R , find $\mathbf{B}$ and $\mathbf{H}$ inside and outside the wire.
3. [25 points] A long non-magnetic cylindrical conductor with inner radius a and outer radius b carries a current I . The current density in the conductor is uniform. Find the magnetic field set up by this current as a function of radius
- (a) inside the hollow space ($r < a$);
 - (b) within the conductor ($a < r < b$);
 - (c) outside the conductor ($r > b$).

Electrodynamics (Part III 50 points)

1. (a) Describe what is the gauge invariance of $\vec{E}$ and $\vec{B}$ fields. (4 points)
(b) Describe what is Coulomb gauge and what is Lorentz gauge. (4 points)
(c) Derive the wave equations for the scalar potential and vector potential when using Lorentz gauge. (8 points)
2. (a) Write down Maxwell's equations in media. (4 points)
(b) By using Maxwell's equations, derive the wave equations of $\vec{E}$ fields for an electromagnetic wave traveling in vacuum. Also prove that the propagation speed of electromagnetic waves in vacuum is light speed. (6 points)
(Hint: $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}$).
(c) Prove that a monochromatic electromagnetic plane wave traveling in vacuum is a TEM wave. (6 points)
(d) Use Maxwell's equations, derive the boundary conditions for E and B fields of electromagnetic waves across the boundary between two linear media (electric permittivity: ϵ_1, ϵ_2 , permeability: μ_1, μ_2 , respectively). (6 points)
(e) Derive the dispersion relation $k(\omega)$ for electromagnetic plane waves transferring in a conducting medium. (6 points)
(f) Give an example: Under what conditions, you can find the propagation of a non-TEM wave (such as TE or TM wave)? (6 points)

1. [15 points] Use Gauss's theorem to prove the following:

- (a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields) (5 points)
- (b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it. (5 points)
- (c) An electric charge distribution produces an electric field

$$\vec{E} = A(1 - e^{-\alpha r}) \frac{\hat{r}}{r^2}$$

where A and α are constants. Find the net charge within the radius $r = 1/\alpha$. (5 points)

2. [14 points] An insulated, spherical, conducting shell of radius R is in a uniform electric field $\vec{E}_0$. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

- (a) if the shell is uncharged; (7 points)
- (b) if the total charge on the shell is Q . (7 points)

3. [21 points] An amount of charge q is uniformly spread out in a layer on the surface of a disc of radius a .

- (a) Use elementary methods based on the azimuthal symmetry of the charge distribution to find the potential at any point on the axis of symmetry. (6 points)
- (b) With the aid of (a) find an expression for the potential at any point $\vec{r}'$ ($|\vec{r}'| > a$) as an expansion in Legendre polynomials. (15 points)

Classical Electrodynamics Part II (50 points)

1. [20 points] A magnetic induction $\vec{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ and with a known $B_z(0, z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z . Show that near the axis and radial components of magnetic induction are approximately

$$B_z(\rho, z) \approx B_z(0, z) - \left(\frac{\rho^2}{4}\right) \left[\frac{\partial^2 B_z(0, z)}{\partial z^2} \right] + \dots$$

$$B_\rho(\rho, z) \approx -\left(\frac{\rho}{2}\right) \left[\frac{\partial B_z(0, z)}{\partial z} \right] + \left(\frac{\rho^3}{16}\right) \left[\frac{\partial^3 B_z(0, z)}{\partial z^3} \right] + \dots$$

2. [15 points] A sphere of radius a carries a uniform surface-charge distribution σ . The sphere is rotated about a diameter with constant angular velocity ω . Find the vector potential both inside and outside the sphere.

Some identities: (i) $\int \frac{\hat{r}'}{|\vec{r} - \vec{r}'|} d\Omega' = f(r)\hat{r},$

(ii) $\int \frac{\cos\gamma}{|\vec{r} - \vec{r}'|} d\Omega' = \sum_{\ell} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} \int P_1(\cos\gamma) P_{\ell}(\cos\gamma) d\Omega',$ where $\cos\gamma = \hat{r} \cdot \hat{r}',$ $r_{<} = \min(r, a),$
 $r_{>} = \max(r, a),$ and P_{ℓ} denote the Legendre polynomials.

3. [15 points] (a) How to define the magnetic dipole moment of a current loop? (use your notations)
 (b) When a magnetic dipole moment ($\vec{m}$) is put in a uniform magnetic induction field ($\vec{B}$), how to derive the potential energy of the system?

Part III: (50 points)

- (1) Electrons oscillate inside plasma under the static magnetic induction $\vec{B}_s = B_0 \hat{z}$ and a plane EM wave, $\vec{E}(t, \vec{x}) = E_0(\hat{x} \pm i\hat{y})e^{-i\omega t + i\vec{k} \cdot \vec{x}}$. Assume that the wave length of the EM wave is much smaller than a given electron's oscillation. You might use following notations (but unnecessarily use each of them) for completing questions: m : electron mass; e : absolute value of electron charge; ω_0 : intrinsic angular frequency; γ : damping constant; N : number per volume.

- Write down the equation of motion (5pts).
- Find a steady-state solution which means "ignoring homogeneous solutions" (10pts).
- Compute the polarization (5pts).
- Obtain permittivity (2pts).
- Is this medium isotropic (1pt)? Give reasons for your answer (2pts).

- (2) A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\vec{\nabla}\Phi$. In this region, it also exists a small localized time-independent current density $\vec{J}(\vec{x})$, which generates a magnetic field $\vec{H}$.

- Show that the momentum of these EM fields, $\vec{P} \equiv \int d^3x [\vec{D} \times \vec{B}]$, can be transformed to $\vec{P} = \frac{1}{c^2} \int d^3x \Phi \vec{J}$ by dropping the surface term (5pts).

• NB: $\epsilon^{ijk} \left(\frac{\partial}{\partial x^j} A \right) B^k = \epsilon^{ijk} \frac{\partial}{\partial x^j} [AB^k] - \epsilon^{ijk} A \frac{\partial}{\partial x^j} B^k$

- Justify in which situation you can ignore the surface term in part (a) (2pts).

- Assuming that the current distribution is localized to a region small compared to the scale of the variation of the electric field, that is, using $\Phi = \Phi_0 - \vec{E}_0 \cdot \vec{x}$. Show $\vec{P} = \frac{1}{c^2} \vec{E}_0 \times \vec{m}$, where $\vec{m}$ is a magnetic dipole moment (8pts).

• NB1: localized source $\Rightarrow \int d^3x J^i(\vec{x}) = 0, \quad \int d^3x [x^k J^\ell(\vec{x}) + x^\ell J^k(\vec{x})] = 0$

• NB2: $\epsilon^{ijk} \epsilon^{klm} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$

- Suppose the current distribution is placed instead in a uniform electric field $\vec{E}_0$ (filling all space). Show that the result in (c) is changed by a surface integral contribution, yielding to $\vec{P} = \frac{2}{3c^2} \vec{E}_0 \times \vec{m}$ (10pts).

• NB: $\int d\Omega \hat{r}^i = 0, \quad \int d\Omega \hat{r}^i \hat{r}^j = \frac{4\pi}{3} \delta^{ij}$

Part III: (50 points)

- (1) Electrons oscillate inside plasma under the static magnetic induction $\vec{B}_s = B_0 \hat{z}$ and a plane EM wave, $\vec{E}(t, \vec{x}) = E_0(\hat{x} \pm i\hat{y})e^{-i\omega t + i\vec{k} \cdot \vec{x}}$. Assume that the wave length of the EM wave is much smaller than a given electron's oscillation. You might use following notations (but unnecessarily use each of them) for completing questions: m : electron mass; e : absolute value of electron charge; ω_0 : intrinsic angular frequency; γ : damping constant; N : number per volume.
- Write down the equation of motion (5pts).
 - Find a steady-state solution which means "ignoring homogeneous solutions" (10pts).
 - Compute the polarization (5pts).
 - Obtain permittivity (2pts).
 - Is this medium isotropic (1pt)? Give reasons for your answer (2pts).
- (2) A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\vec{\nabla}\Phi$. In this region, it also exists a small localized time-independent current density $\vec{J}(\vec{x})$, which generates a magnetic field $\vec{H}$.
- Show that the momentum of these EM fields, $\vec{P} \equiv \int d^3x [\vec{D} \times \vec{B}]$, can be transformed to $\vec{P} = \frac{1}{c^2} \int d^3x \Phi \vec{J}$ by dropping the surface term (5pts).
 - NB: $\epsilon^{ijk}(\frac{\partial}{\partial x^j} A)B^k = \epsilon^{ijk} \frac{\partial}{\partial x^j} [AB^k] - \epsilon^{ijk} A \frac{\partial}{\partial x^j} B^k$
 - Justify in which situation you can ignore the surface term in part (a) (2pts).
 - Assuming that the current distribution is localized to a region small compared to the scale of the variation of the electric field, that is, using $\Phi = \Phi_0 - \vec{E}_0 \cdot \vec{x}$. Show $\vec{P} = \frac{1}{c^2} \vec{E}_0 \times \vec{m}$, where $\vec{m}$ is a magnetic dipole moment (8pts).
 - NB1: localized source $\implies \int d^3x J^i(\vec{x}) = 0, \quad \int d^3x [x^k J^\ell(\vec{x}) + x^\ell J^k(\vec{x})] = 0$
 - NB2: $\epsilon^{ijk} \epsilon^{klm} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$
 - Suppose the current distribution is placed instead in a uniform electric field $\vec{E}_0$ (filling all space). Show that the result in (c) is changed by a surface integral contribution, yielding to $\vec{P} = \frac{2}{3c^2} \vec{E}_0 \times \vec{m}$ (10pts).
 - NB: $\int d\Omega \hat{r}^i = 0, \quad \int d\Omega \hat{r}^i \hat{r}^j = \frac{4\pi}{3} \delta^{ij}$

Electrodynamics (Part I: Electrostatics 50 points).

1. (a) From the Green's theorem (Green's second identity):

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3x = \oint_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da,$$

where ϕ and ψ are arbitrary scalar fields, show that the solution of a Dirichlet boundary-value problem is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho_f(\vec{r}') G_D(\vec{r}, \vec{r}') d\tau' - \frac{1}{4\pi} \oint_S \Phi(\vec{r}') \frac{\partial G_D}{\partial n'} da'.$$

To do so, what are the required conditions to choose a suitable Green's function G_D ? **(8 points)**

- (b) As shown in Fig. 1, the conducting hemi-spherical shell of radius a and the infinite conducting plane are grounded. A point charge q ($q>0$) is placed on the z axis and the distance from the charge to the center of the hemi sphere is D .

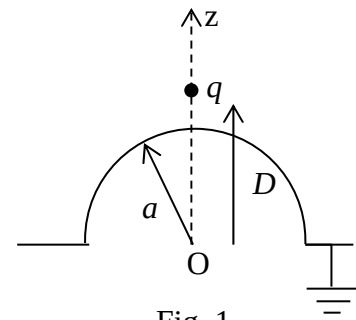


Fig. 1

- (i) Find the positions and magnitudes of all image charges. **(6 points)**
(ii) Write down the appropriate Green function $G(\vec{r}, \vec{r}')$ for Dirichlet boundary value problems in the region $z>0$ and $r>a$. **(5 points)**

2. As shown in Fig. 2, the surface charge density on a thin spherical shell with radius b is $\sigma_l(\theta') = \sigma_l \cos^2 \theta'$. Find the electric potential inside ($r<b$) and outside ($r>b$) the spherical shell. **(15 points)**

Legendre polynomials: $P_0(x) = 1$; $P_1(x) = x$;

$$P_2(x) = (3x^2 - 1)/2; P_3(x) = (5x^3 - 3x)/2; \dots$$

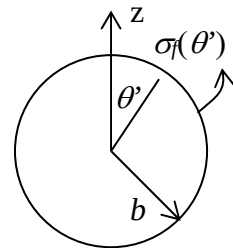


Fig. 2

3. As shown in Fig. 3, charge Q was uniformly distributed on a spherical conductor surface (radius a). The conductor was surrounded by a dielectric shell of inner radius a and outer radius b . The electric permittivity of the linear dielectric is ϵ .

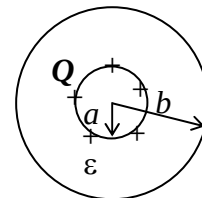


Fig. 3

- (a) Find the electric fields $\vec{E}$ at $b>r>a$ and $r>b$. **(6 points)**
(b) Find the polarization surface bound charges at $r=a$ and $r=b$. **(4 points)**
(c) Find the electric force per unit volume on the dielectric (Express your answer by using the position vector from the sphere center: $\vec{r}$). **(6 points)**

Part II: Magneto-statics (50 points)

1.

For a line current distribution $\mathbf{I}(\mathbf{r}')$, Biot –Savart law can be written as

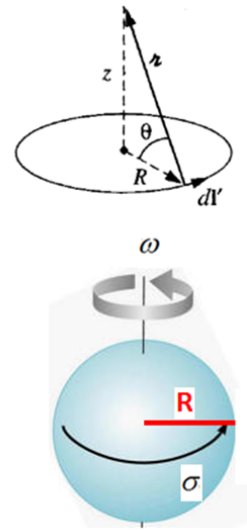
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times \hat{\mathbf{r}}}{r^2} d\mathbf{l}' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$$

- (1) show that the magnetic field at a distance above the center of a circular loop of radius R , and carries a current I is

$$\mathbf{B} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

- (2) using the results of (1), show that the magnetic field at the center of a sphere of radius R , carrying a uniform surface charge σ and spinning at angular velocity ω is

$$\mathbf{B} = \frac{2}{3} \mu_0 R \sigma \omega \hat{\mathbf{z}} \quad (\text{or } \frac{\mu_0 Q \omega}{6\pi R} \hat{\mathbf{z}}, \sigma = \frac{Q}{4\pi R^2})$$



2.

For surface current distributions, the magnetic vector potential can be written as

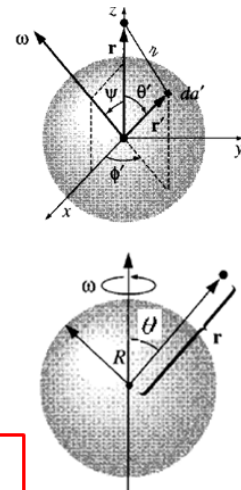
$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} da', \quad \text{where } \mathbf{K} \text{ is the surface current}$$

- (a) Show that the magnetic vector potential **inside** a sphere of radius R , carrying a uniform surface charge σ , spinning at angular velocity ω , is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R \sigma}{3} (\boldsymbol{\omega} \times \mathbf{r}) = \frac{\mu_0 R \omega \sigma}{3} r \sin \theta \hat{\boldsymbol{\phi}},$$

- (b) Show that the magnetic field inside the sphere is

$$\mathbf{B} = \frac{2\mu_0 R \omega \sigma}{3} (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}) = \frac{2}{3} \mu_0 R \sigma \omega \hat{\mathbf{z}}$$



Hint:
$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}} du = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rru} \Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2} [(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)]$$

$$= 2r/3R^2 \quad \text{If } r \leq R$$

$$\nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial}{\partial \phi} A_\theta \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} A_r - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial \theta} A_r \right] \hat{\boldsymbol{\phi}}$$

Part III: Electromagnetic Waves (50 points)

- (1) Use the Lorentz force law and Maxwell equations to work out the dimension of the following quantities in terms of charge q , mass m , length ℓ and time t :
 - (a) electric displacement $\vec{D}$, (b) current density $\vec{J}$, (c) magnetic field $\vec{H}$,
 - (d) electric field $\vec{E}$, (e) magnetic induction $\vec{B}$, (f) poynting vector $\vec{S}$,
 - (g) electromagnetic momentum $\vec{D} \times \vec{B}$, (h) Maxwell tress tensor M_{ij} .
- (2) Use the Lorentz force law and four Maxwell equations to point out
 - (a) how the quantities in (1) change under parity transformation $\vec{x} \rightarrow \vec{x}' = -\vec{x}$;
 - (b) and how they change under time reversal $t \rightarrow t' = -t$.
- (3) Use spatial rotation, parity and time reversal (a) to pick up allowed terms of the following model for polarization (c_1, c_2 are scalars; $\alpha^{ij}, \beta^{ijk}, \gamma^{ijk}$ etc. are tensors):

$$\frac{1}{\epsilon_0} P^i = \alpha^{ij} E^j + \beta^{ijk} E^j E^k + \gamma^{ijk} E^j B^k + \delta^{ijk} B^j B^k + c_1 (\vec{E} \times \vec{B})^i + c_2 (\vec{E} \times \vec{B})^{\ddot{i}}$$
 - (b) For the allowed terms, what are generic expressions for these tensors?
- (4) (a) Obtain the 2nd order differential equation for magnetic induction $\vec{B}(t, \vec{x})$ in medium with permeability μ and permittivity ϵ from $\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$ and $\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0$ (assume medium is linear, isotropic and homogeneous).
 - (b) Show the sources, $\rho(t, \vec{x}) = -\theta(t) \vec{p} \cdot \vec{\nabla} \delta^3(\vec{x})$ and $\vec{J}(t, \vec{x}) = \vec{p} \delta(t) \delta^3(\vec{x})$, satisfy current conservation. (c) Compute the magnetic induction with the sources identified in (b) using retarded Green's function $\frac{\delta(t-t' - \frac{1}{c} \|\vec{x} - \vec{x}'\|)}{\|\vec{x} - \vec{x}'\|}$.

Part III (50 points)

(1) Prove the Poynting's theorem (15%)

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}. \quad (1)$$

U is the electromagnetic field energy density, and $\mathbf{S}$ is the Poynting vector.

(2) Prove that the propagation velocity of electromagnetic wave in any materials is slower than the light velocity in vacuum (15%).

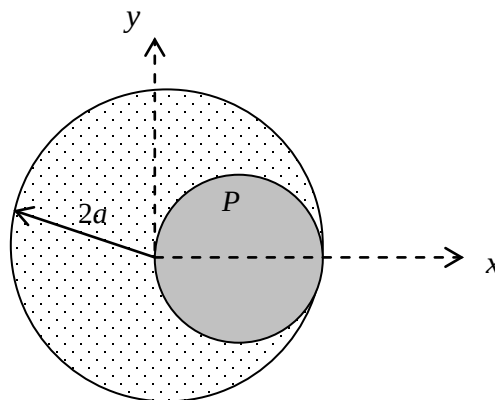
(3) Consider a square waveguide of length a enclosed by a perfect conductor, in which it is composed of the non-dissipative materials with dielectric constant ϵ and magnetic permeability μ . (a) Derive the Helmholtz equations for the electric and magnetic fields from the Maxwell equations (10%), and (b) calculate the axial magnetic field of the threshold TE wave (10%).

1. **(10 Points)** Assuming that the electric field intensity $\mathbf{E}(x,y,z) = 10x\mathbf{i} + 10y\mathbf{j} + 10\mathbf{k}$ (V/m), find the total electric charge contained inside a cubical volume of 20 cm on a side centered symmetrically at the origin.

2. **(15 Points)** A point charge q is embedded at the center of a sphere of linear dielectric material (electric permittivity ϵ and radius a). (a) Find the electric displacement $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization $\mathbf{P}$ inside the sphere. (b) What is the polarization surface charge density?

3. **(10 Points)** An electric dipole $\mathbf{p}$ with a magnitude of 2×10^{-27} C-m is initially located at an origin along the z -axis. (a) Determine the rotational torque on this dipole when an electric field $\mathbf{E} = 5x + 10y$ (V/m) is applied. (b) How much energy changes if $\mathbf{p}$ finally points to the direction of $\mathbf{E}$?

4. **(15 Points)** A sphere with the dotted region of radius $2a$ has a uniform charge density ρ_0 while the uniform charge density inside the small spherical part of radius a is $2\rho_0$. Find the magnitude and direction of $\mathbf{E}$ field at the position $P(a, a/2, 0)$.

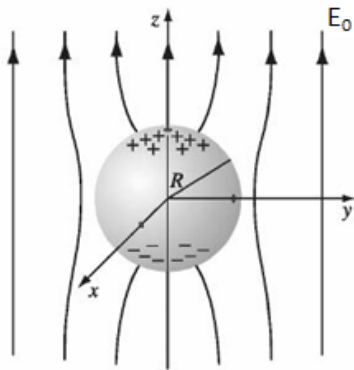


Part II (50 points)

(1) A dielectric sphere of radius R with dielectric constant ϵ/ϵ_0 placed in an initially uniform electric field, which at large distances from the sphere is directed along the z axis and has magnitude of E_0 , as shown below. Both inside and outside there are no free charges.

(a) Find the electric potential and electric field everywhere. (15 points)

(b) Calculate the induced polarization and surface charge density of the dielectric sphere. (10 points)



(2) A long cylindrical conductor (of radius a) has an off-center cylindrical hole (of radius $a/2$) down its full length, as shown in the following figure. If a current I flows through the conductor into the page, show that

(a) The volume current density flow through the conductor is

$$J = \frac{4I}{3\pi a^2} \quad (5 \text{ points})$$

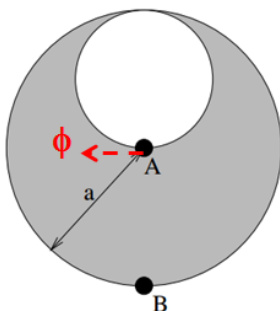
(b) the strength and direction of the magnetic field in point A is

$$\mathbf{B} = \frac{\mu_0 I}{3\pi a} (-\hat{\phi}) \quad (10 \text{ points})$$

(c) the strength and direction of the magnetic field in point B is

$$\mathbf{B} = \frac{5\mu_0 I}{9\pi a} \hat{\phi} \quad (10 \text{ points})$$

(hint: using superposition principle and Ampere's law)



Part III (50 points)

(1) By using the Maxwell equations, the wave equations for the fields with given charge (ρ) and current ($\mathbf{J}$) densities are described by (10 points)

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{-1}{\epsilon_0} (-\nabla \rho - \frac{1}{c^2} \frac{\partial \mathbf{J}}{\partial t})$$

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J}$$

The electromagnetic fields for the above equations are (10 points)

$$\mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3 x' \left\{ \frac{\mathbf{R}}{R^3} [\rho(\mathbf{x}', t')]_{\text{ret}} + \frac{\mathbf{R}}{cR^2} \left[\frac{\partial \rho(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{cR^2} \left[\frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \right\}$$

and

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3 x' \left\{ [\mathbf{J}(\mathbf{x}', t')]_{\text{ret}} \times \frac{\mathbf{R}}{R^3} + \left[\frac{\partial \mathbf{J}(\mathbf{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\mathbf{R}}{cR^2} \right\}$$

where $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $R = |\mathbf{x} - \mathbf{x}'|$.

(2) The frequency-dependent dielectric constant $\epsilon(\omega)/\epsilon_0$ is an analytic function of ω in the upper half – plane. Show that its real and imaginary parts satisfy the Kramers-Kronig relations (15 points).

(3) We consider the propagation of TM waves in a rectangular waveguide with length a and width b , as shown in Fig. 1. The metallic waveguide is filled with dielectric constant ϵ and permeability μ . Calculate the electromagnetic fields, the cutoff frequency, and the field energy per unit length (15 points).

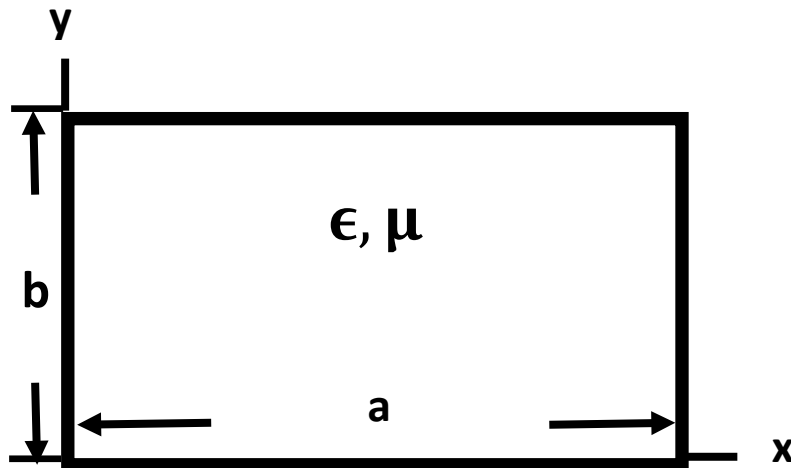


Fig. 1

Electrodynamics Part I (50 points)

1. Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R .

[7 pts]

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b . [7 pts]

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R . [6 pts]

2. In Figure 2, find the potential in the region, $0 \leq x \leq a$, $y \geq 0$, under the boundary conditions that $\Phi = 0$ at $x = 0$ and $x = a$, while $\Phi = V$ at $y = 0$ for $0 \leq x \leq a$ and $\Phi \rightarrow 0$ for large y . [10 pts]

3. Three point charges ($q, -2q, q$) are located in a straight line with separation a and with the middle charge $-2q$ at the origin of a grounded conducting spherical shell of radius b , as indicated in Fig. 3.

(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \rightarrow 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates. [10 pts]

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \rightarrow 0$,

$$\Phi(r, \theta, \varphi) \rightarrow \frac{Q}{2\pi\epsilon_0 r^3} \left(1 - \frac{r^5}{b^5}\right) P_2(\cos \theta) \quad [10\text{pts}]$$

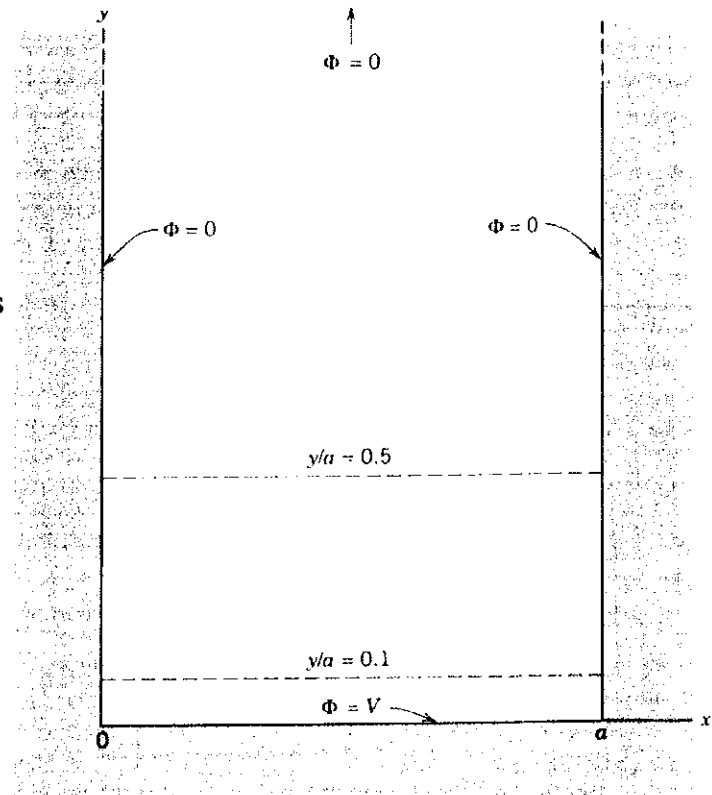


FIG 2

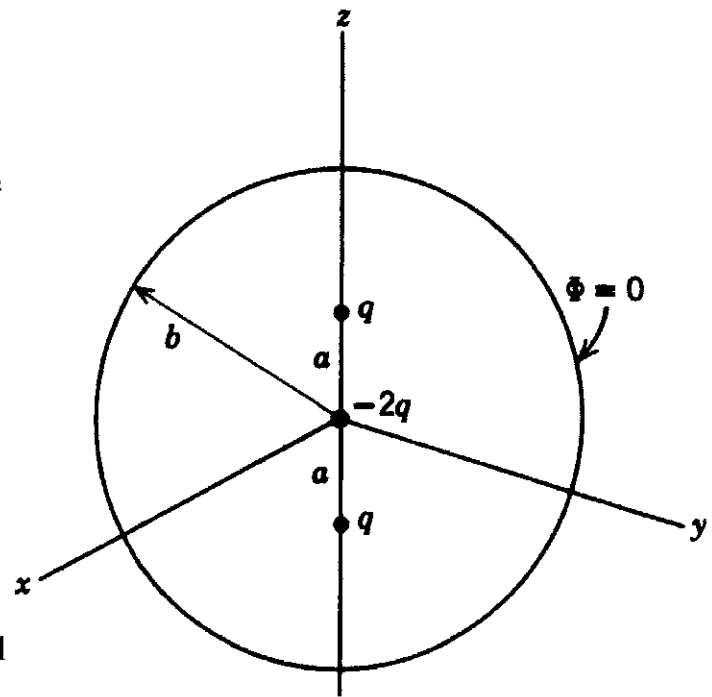
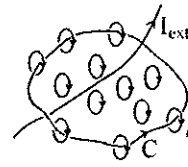


FIG 3

Electrodynamics - Part 2 (50 points)

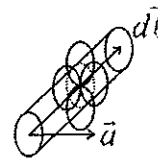
Problem 1: (15 points)

In medium, we can represent each molecule by a current loop and its dipole moment by $\vec{m} = i\vec{a}$. Assume the number of molecules (loops) per unit volume is N , then the magnetic dipole density is $\vec{M} = Ni\vec{a}$.



By applying the Ampere's law in vacuum $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \vec{J} \cdot d\vec{A} = \mu_0 I$

(S is any surface bounded by C), show that in medium



$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 (I_{ext} + \oint_C \vec{M} \cdot d\vec{\ell})$, where I_{ext} is the external current. (Note that

this is the integral form of $\vec{\nabla} \times (\vec{B}/\mu_0 - \vec{M}) = \vec{J}_{ext}$.)

Problem 2: (15 points)

The mutual inductances between two current loops are equal, i.e. $M_{12} = M_{21} \equiv M$.

Now we have a very small and a large current loops which are originally far apart, each maintained at fixed currents i & I respectively by batteries. The small loop, with magnetic dipole $\vec{m} = i\vec{a}$, is then brought to the vicinity of the large loop, where the magnetic field is $\vec{B}(\vec{r})$. By carefully considering the energy supplied by the batteries to maintain the currents and the work done by the torque $\vec{m} \times \vec{B}(\vec{r})$, show that the total change in the magnetic energy is $+\vec{m} \cdot \vec{B}$ (not $-\vec{m} \cdot \vec{B}$).

Problem 3: (10 points)

It is well known that the magnetic field by an infinitely long and dense solenoid carrying a current is uniform $\vec{B} = B\hat{z}$ inside, and zero outside. Assuming the solenoid is circular, ^{with a radius R} , find the vector potential $\vec{A}$ everywhere (inside and outside) in terms of B & r , where r is the distance from the central axis of solenoid. In what gauge is your $\vec{A}$?

Problem 4: (10 points)

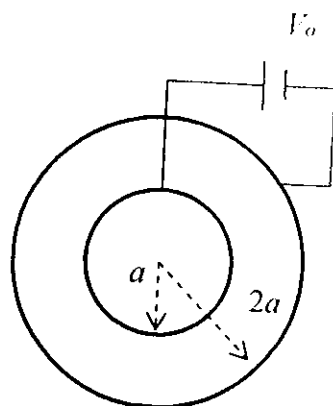
Derive the conservation of charge from the Maxwell's eqs.

Electrodynamics (Part III, 50 points) 2016-10-21

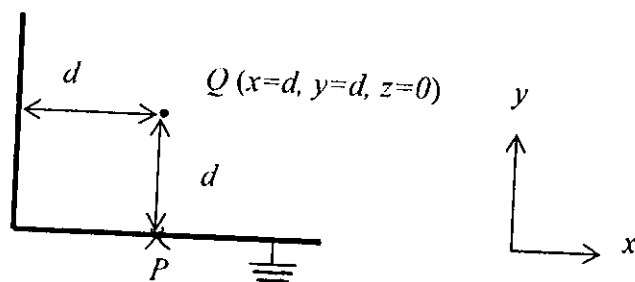
1. Assuming that a monochromatic point source of N (Watt) is sending out a linearly polarized spherical wave into every direction.
 - (a) Please write down the wave function of the spherical wave and show that your solution satisfies the wave equation. **(15 points)**
 - (b) What is the momentum density of this spherical wave? **(8 points)**
 - (c) Calculate the root mean square electric field at 1 meter away from the point source. **(7 points)**

2. A monochromatic electromagnetic plane wave is propagating toward the z -direction in a conductor, where the polarization of the electromagnetic plane wave is along x -direction. The wave number of the plane wave in a conductor is a complex number, i.e., $\tilde{k} = k + i\kappa$.
 - (a) Determine the amplitude and the phase relation between the E -field and the B -field of the plane wave. **(10 points)**
 - (b) Express B -field of the plane wave by the E -field amplitude E_0 and E -field phase δ_E . **(10 points)**

1. (25 Points) Assuming that the space between the inner and outer conductors of a concentric sphere is filled with a uniform charge cloud having a volume charge density $\rho(r) = K/r$ for $a < r < b = 2a$, where a and b are the radii of the inner and outer conductors respectively. The inner conductor is maintained at a constant potential V_0 , while the outer one is grounded. Determine the electric potential distribution in the region $a < r < b$ by solving the Poisson's equation.



2. (25 Points) A positive point charge Q is located at a distance d from two grounded perpendicular conducting half-planes, as shown below.
- Find the electric field intensity E (magnitude and direction) at the point $P(x = d, y = 0, z = 0)$.
 - Determine the force (magnitude and direction) on Q caused by the charges induced on the planes.



*Useful formula:

In spherical coordinates:

$$\nabla^2(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Classical Electrodynamics: II (50%)

- (1) There are two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ at $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. Determine the arrangement with the lowest interaction energy and its energy. (10%)
- (2) The electric dipole moment $\mathbf{p}$ is situated at the center of a sphere with radius R . Calculate the volume integral of the electric field over the sphere. (15%) Also noticed that there are useful mathematical equations in appendix.
- (3) (a) Calculate the magnetic scalar potential Φ_M for a sphere of radius a , with a uniform permanent magnetization $\mathbf{M}$. (10%) (b) Now, the sphere is a linear magnetic substance, and its magnetic permeability is μ . Moreover, it exists in a uniform magnetic induction $\mathbf{B}_0$. Find the relation between $\mathbf{M}$ and $\mathbf{B}_0$. (5%)
- (4) A very long cylinder of radius a carries a current I . Calculate the magnetic inductance per unit length inside the cylinder. (10%)

Appendix: useful mathematical equations

$$\int_{r \leq R} \nabla \Phi(\mathbf{x}) d^3x = \int_{r=R} \Phi(\mathbf{x}) R^2 \mathbf{n} d\Omega. \quad (1)$$

$$\int \cos \gamma \mathbf{n} d\Omega = \frac{4\pi \mathbf{n}'}{3}. \quad (2)$$

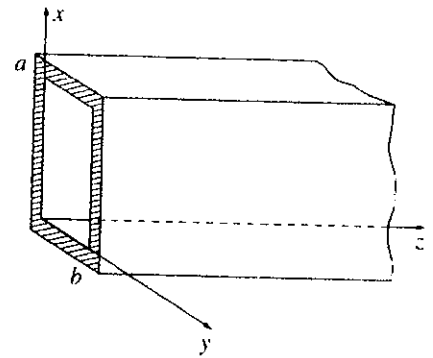
$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \gamma). \quad (3)$$

Electrodynamics (Part III, 50 points) 2016-03-25

1. A monochromatic electromagnetic plane wave is propagating toward the z-direction in vacuum, where the polarization of the electromagnetic plane wave is along x-direction and the wavelength of the plane wave is λ .
 - (a) Write down the expression of the E-field of the plane wave. (3 points)
 - (b) Express B-field of the plane wave by the E-field amplitude and E-field phase. (3 points)
 - (c) Find the long-time-averaged value of three quantities of the plane wave: the energy density, Poynting vector, and momentum density. (9 points)
2. Write down the integral form of two important conservation laws of electromagnetism, i.e. the conservation law of electromagnetic energy and the conservation law of electromagnetic momentum, respectively. Clearly address the physical meanings of each term in two expressions. (15 points)

3. Suppose we have a wave guide of rectangular shape, with height a and width b ($a > b$). It is a hollow pipe and the wave guide is composed of perfect conductor. In the metal pipe, the longitudinal components E_z and B_z of the guided waves should fulfill the equations,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0,$$
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] B_z = 0.$$



Where the other components, E_x , E_y , B_x , and B_y , could be determined by the longitudinal components E_z and B_z of the guided waves, i.e.,

$$(i) \quad E_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right),$$

$$(ii) \quad E_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right),$$

$$(iii) \quad B_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right),$$

$$(iv) \quad B_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right).$$

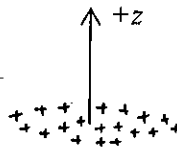
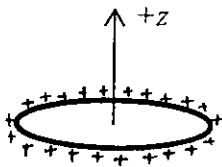
We are interest in the TM waves in the rectangular wave guide.

- (a) Find the TM waves in the rectangular wave guide. (8 points)
- (b) Find the lowest cutoff frequency of TM modes. (4 points)
- (c) Find the phase and group velocity of the TM waves. (8 points)

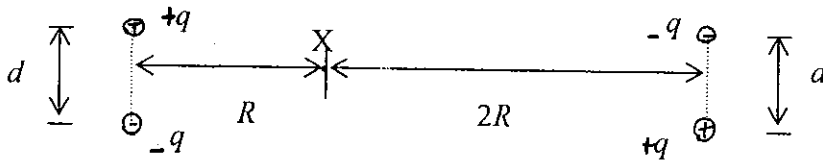
Electrodynamics (Part I, 50 points)

2015-10-16

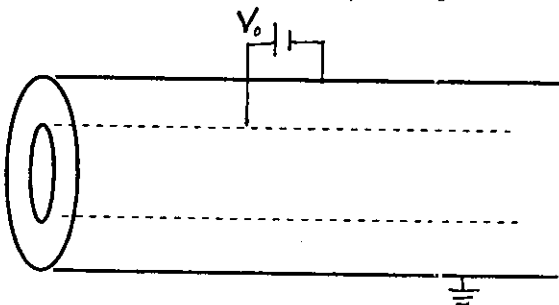
1. (15 Points) (a) Calculate the electric potential $V(z)$ at a point z along the axis of a charged ring of radius a which contains a total charge Q . (b) Find the z -component of the electric field (E_z) at the point z . (c) Where is the maximum E_z for this charged ring? (d) From the result of (a), calculate the electric potential $V(z)$ at a point z along the axis of a charged disk of radius a which contains a total charge Q . (e) Find the z -component of the electric field (E_z) at the point z . (f) Where is the maximum E_z for this charged disk?



2. (20 Points) (a) Evaluate the electrostatic energy for the configuration of four charged particles shown below. (b) Compute the electric potential at the position X. (c) Calculate the electric field E (magnitude and direction) at point X. (d) If a charged particle $-Q$ is placed at the position X, what is the electric force experienced by that particle? You should provide the magnitude and direction for the force in your answer. (e) Compute the electric field intensity E at position X as $R \gg d$. Express your answer in terms of the definition of electric dipole p .



3. (15 Points) Assuming that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density charge $\rho(r) = A/r$ for $a < r < 2a$, where a and $2a$ are the radii of the inner and outer conductors respectively. The inner conductor is maintained at a constant potential V_0 while the outer conductor is grounded. Determine the potential distribution in the region $a < r < 2a$ by solving the Poisson's equation with numerical values of $a = 2$ m, $A = \epsilon_0 \ln 2$ C/m², and $V_0 = \ln 2$ Volt. (Here $\epsilon_0 = 1/(36\pi \times 10^9)$ F/m)



In cylindrical coordinates,

$$\nabla^2(r, \phi, z) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Electrodynamics (Part II 50 points).

1. A solenoid consists of n turns per unit length wrapped around a cylindrical tube of radius a and carries current I (Fig. 1).
 - (a) As shown in Fig. 1, find the magnetic field at point P on the axis of the solenoid. Express your answer in terms of θ_1 and θ_2 . (10 pts)
 - (b) If the solenoid is infinite long, find the field on the axis of the infinite solenoid. (5 pts)
 - (c) Find the vector potential $\vec{A}$ inside and outside an infinite solenoid. (10 pts)

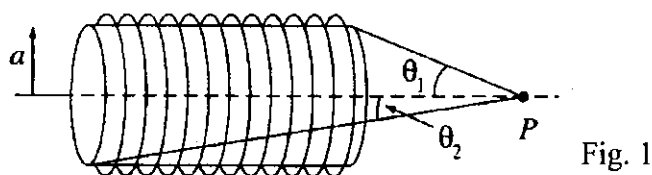


Fig. 1

2. (a) Is $\vec{\nabla} \cdot \vec{H}$ always equal to zero in magnetostatic cases? If not, provide an example to show $\vec{\nabla} \cdot \vec{H} \neq 0$. (5 pts)
- (b) Can you always express $\vec{H}$ as $-\vec{\nabla} \cdot V_M$? where V_M is a magnetic scalar potential. If not, please describe the required conditions. (5 pts)
- (c) As shown in Fig. 2, a very large piece of hard magnetic material has magnetization $\vec{M}$. The B-field inside the material is $\vec{B}_0$, and the H-field inside the material is $\vec{H}_0$. Now a cavity is hollowed out of the material. Find the B- and H- fields at the center of the cavity for three cases: (10 pts)
 - (i) The cavity is a small sphere.
 - (ii) The cavity is a long needle running parallel to $\vec{M}$.
 - (iii) The cavity is a thin wafer perpendicular to $\vec{M}$.

Express the answer of B-fields in terms of $\vec{B}_0$ and $\vec{M}$, and H-fields in terms of $\vec{H}_0$ and $\vec{M}$.

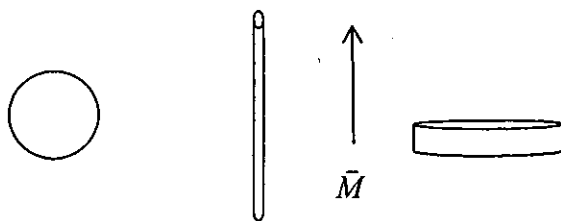


Fig. 2

3. Can a static magnetic field exist inside a perfect conductor? Explain why or why not. (5 pts)

Electrodynamics (Part III, 50 points) 2015-10-16

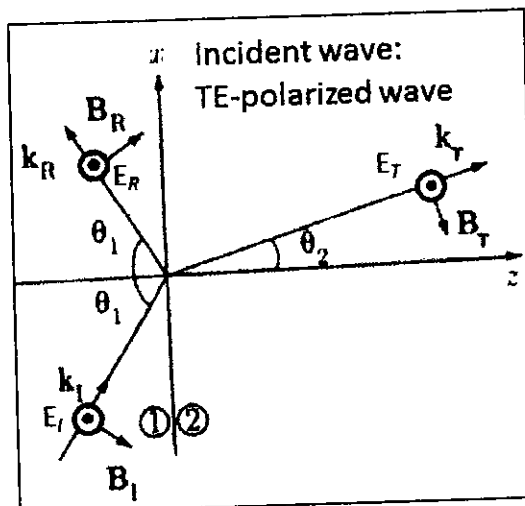
1. The Maxwell equations can be rephrased as two equations,

$$\begin{cases} \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0} \rho \\ \left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \mathbf{J} \end{cases}, \dots\dots\dots (*)$$

which contains all the information in Maxwell's equations.

- (a) What is the gauge transformation? (10 points)
 (b) Show that in the Lorentz gauge the addressed two equations (*) can be rephrased in the form of two independent inhomogeneous wave equations. (10 points)

2. Referring to the figure, an electromagnetic wave of polarization perpendicular to the plane of incidence is obliquely impinging on the boundary. Medium 1 and 2 are both dielectrics. The incident, reflected, transmitted waves are represented as following:



$$\begin{cases} \tilde{\mathbf{E}}_I = \tilde{E}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I = \frac{1}{v_1} \tilde{E}_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} (-\cos \theta_1 \hat{\mathbf{x}} + \sin \theta_1 \hat{\mathbf{z}}); \\ \tilde{\mathbf{E}}_R = \tilde{E}_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} (\cos \theta_1 \hat{\mathbf{x}} + \sin \theta_1 \hat{\mathbf{z}}); \\ \tilde{\mathbf{E}}_T = \tilde{E}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} (-\cos \theta_2 \hat{\mathbf{x}} + \sin \theta_2 \hat{\mathbf{z}}); \end{cases}$$

- (a) Write down boundary conditions of E-field and B-field. (10 points)
 (b) Impose the boundary conditions, and obtain the Fresnel equations for E_{0R} and E_{0T} . (20 points)